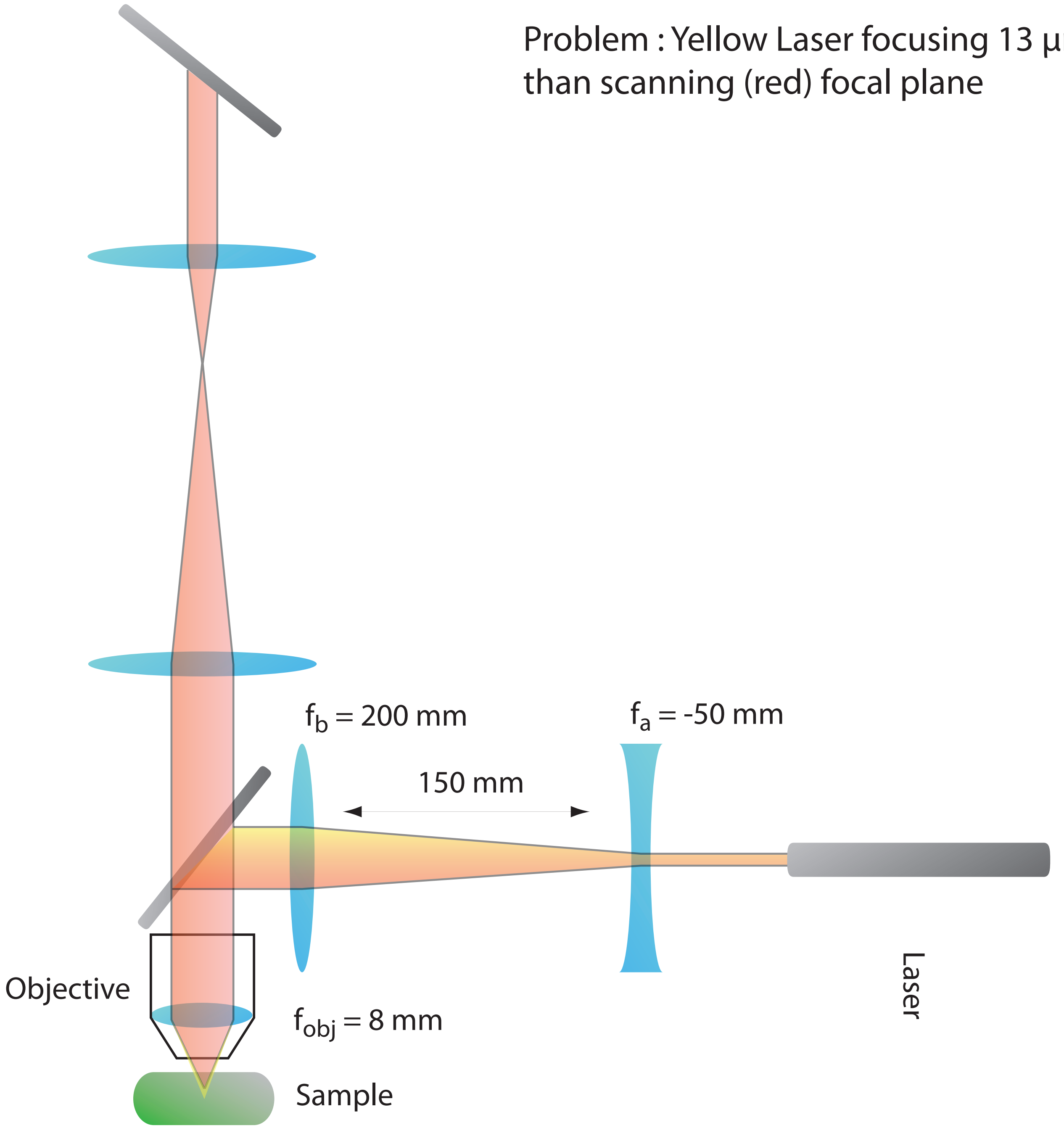
Ray Matrices : a practical application

Problem : Yellow Laser focusing 13 µm deeper than scanning (red) focal plane



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Ray Matrices

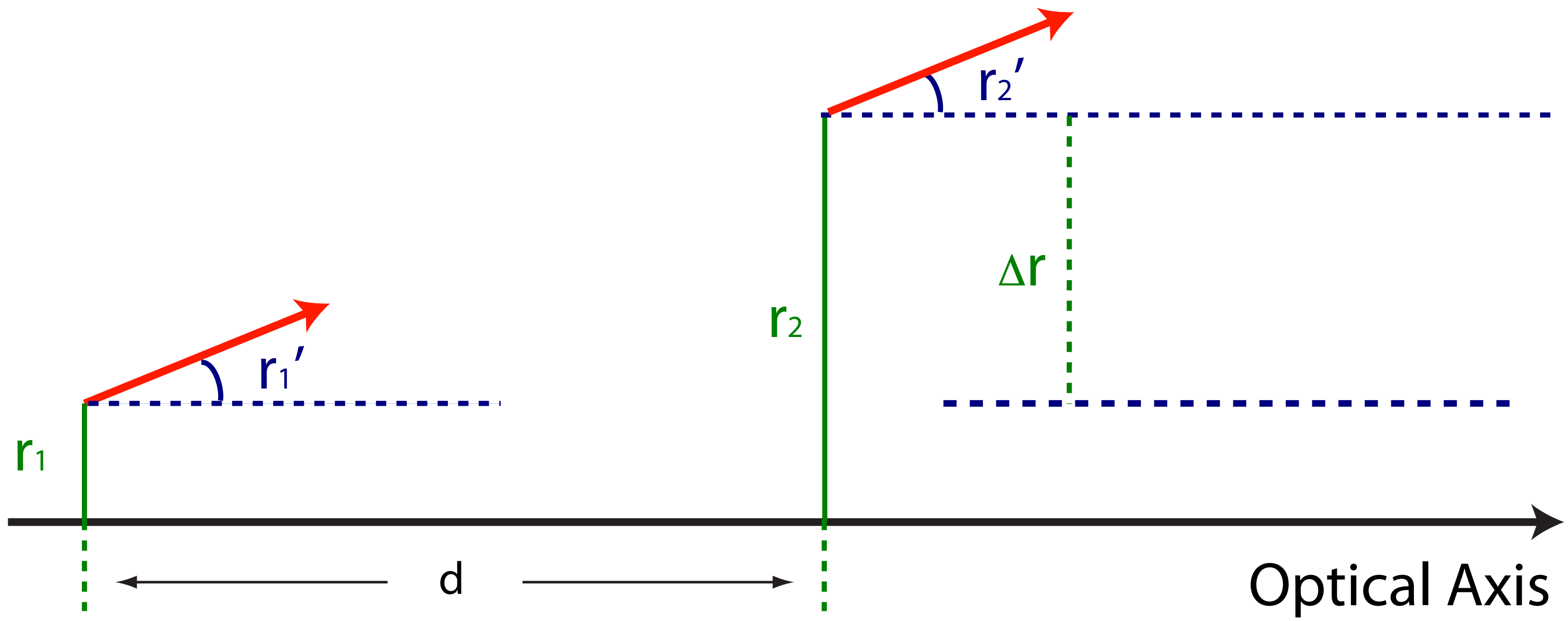
ABCD Matrix for propagation

r = height from optical axis

r' = tangent of angle ray makes with optical axis

Ray Matrices

ABCD Matrix for propagation



$$r_2 = r_1 + \Delta r$$
$$= r_1 + d \cdot \tan(\theta_1)$$
$$= 1 \cdot r_1 + d \cdot r_1'$$

$$r_2' = r_1'$$
$$= 0 \cdot r_1 + 1 \cdot r_1'$$

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Ray Matrices

Review of Matrix Algebra

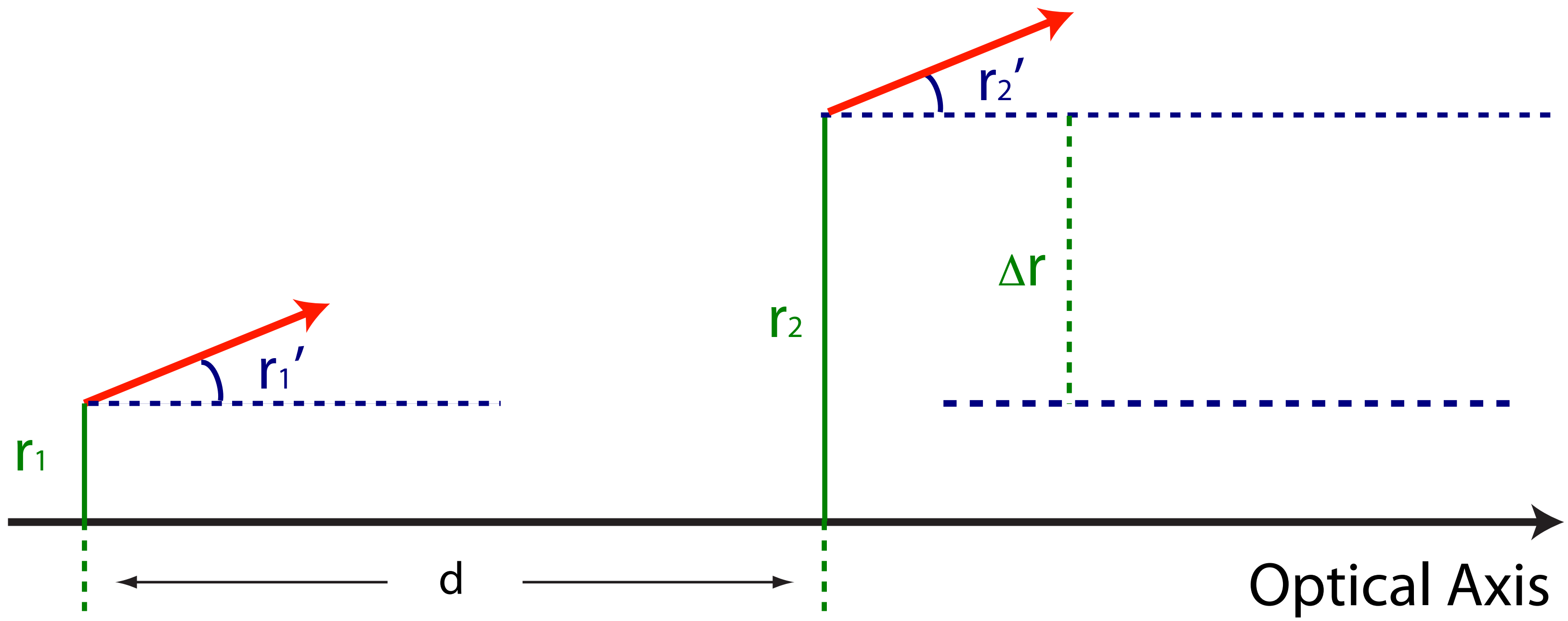
$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_1 \\ r'_1 \end{bmatrix}$$

$$r_2 = A \cdot r_1 + B \cdot r'_1$$

$$r'_2 = C \cdot r_1 + D \cdot r'_1$$

Ray Matrices

ABCD Matrix for propagation



$$
\begin{aligned}
r_2 &= r_1 + \Delta r \\
&= r_1 + d \cdot \tan(\theta_1) \\
&= 1 \cdot r_1 + d \cdot r_1'
\end{aligned}
$$

$$
\begin{aligned}
r_2' &= r_1' \\
&= 0 \cdot r_1 + 1 \cdot r_1'
\end{aligned}
$$

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\text{propagation}} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}
$$

Approximations

Brutalizing optics into 4 limiting regimes

- Ray (Geometric Optics) : $\lambda \rightarrow 0$
- Paraxial Approximation : $\theta \ll \pi / 2$
- Thin Lens Approximation : lens thickness $\rightarrow 0$
- Lossless Approximation : scatter, absorption $\rightarrow 0$

Ray Matrices

ABCD Matrix for flat interfaces (Snell's Law)

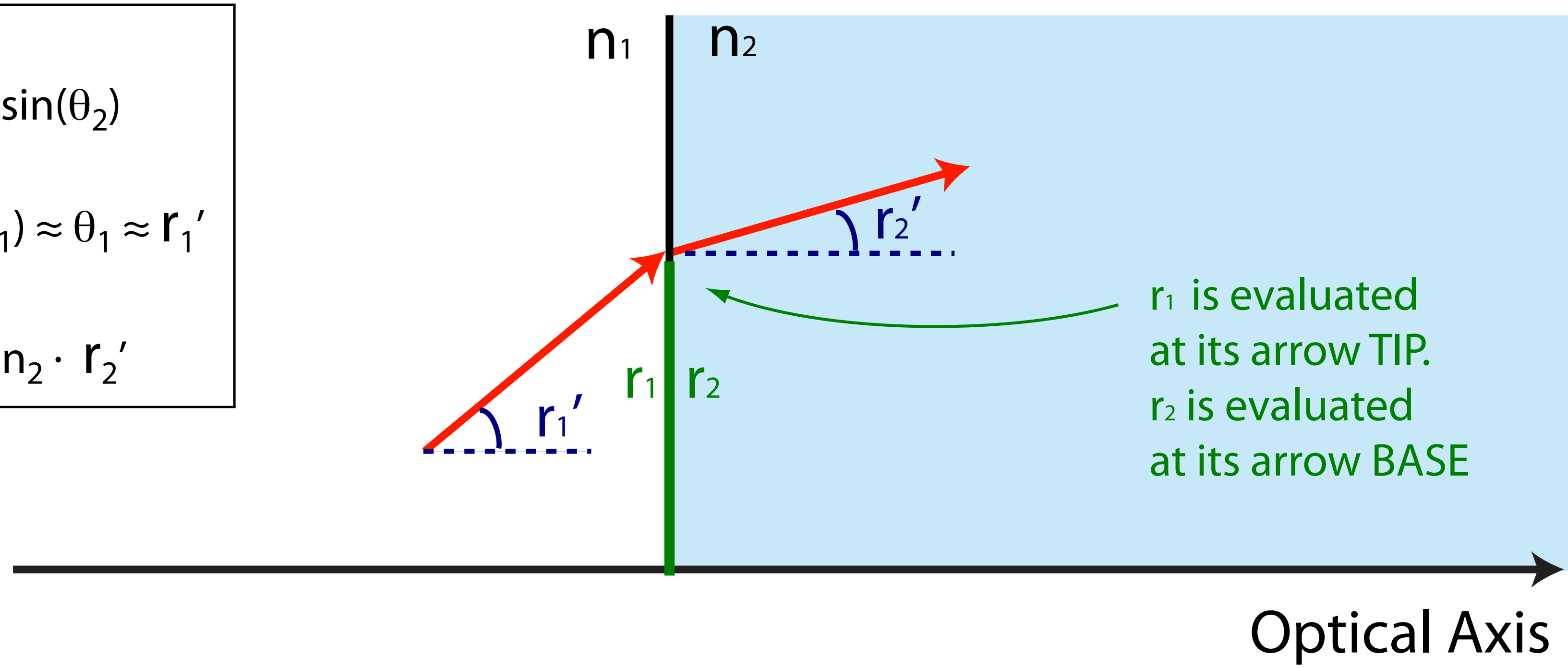
Snell's Law:

$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

but:

$\sin(\theta_1) \approx \tan(\theta_1) \approx \theta_1 \approx r_1'$

$\Rightarrow \; n_1 \cdot r_1' = n_2 \cdot r_2'$



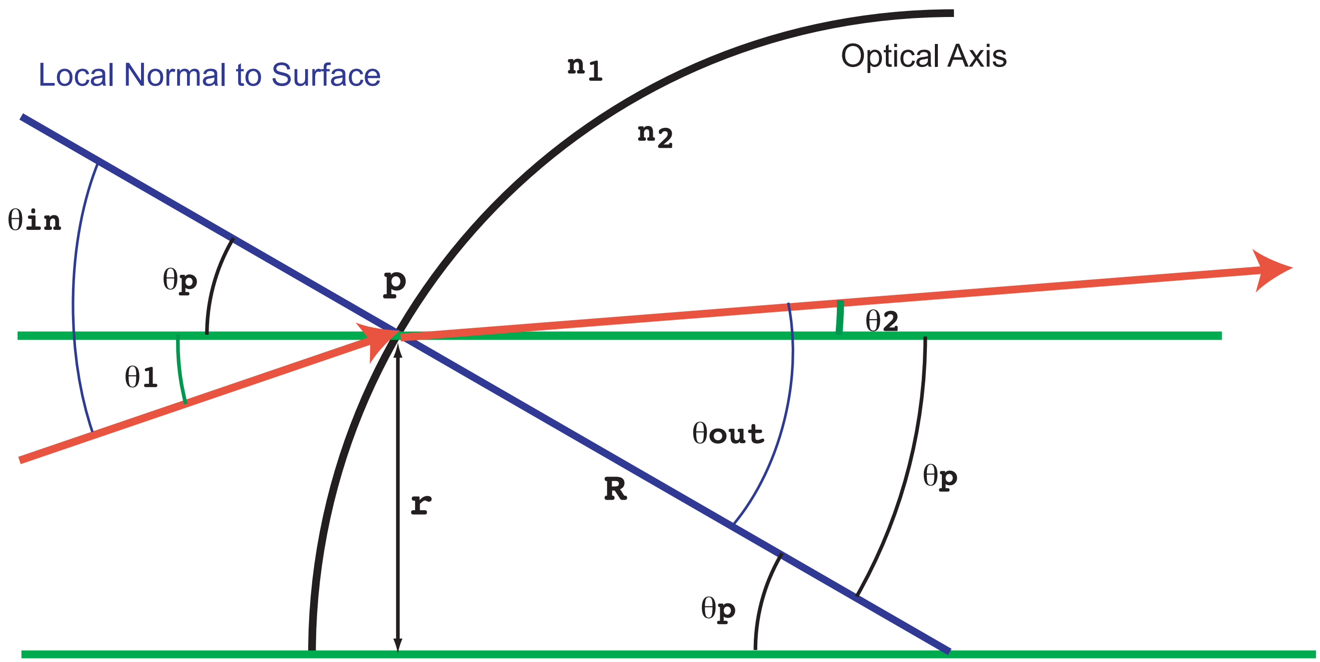
$$r_2 = r_1 = 1 \cdot r_1 + 0$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\text{flat interface}} = \begin{bmatrix} 1 & 0 \\ 0 & (n_1/n_2) \end{bmatrix}$$

$$r_2' = (n_1/n_2) \cdot r_1' = 0 \cdot r_1 + (n_1/n_2) \cdot r_1'$$

Ray Matrices

ABCD Matrix for curved interface



Input ray strikes point P on the spherical surface

R = radius of curvature of the spherical surface

θ_p = angle from center of curvature up to the point P

θ_1 = angle, before refraction, of light ray with respect to the optical axis

θ_{in} = Snell's law input angle of light ray with respect to the local surface normal

θ_{out} = Snell's law output angle of light ray with respect to the local surface normal

θ_2 = angle, after refraction, of light ray with respect to the optical axis

$r = r_1 = r_2$ = vertical height from the optical axis of the ray at point of contact

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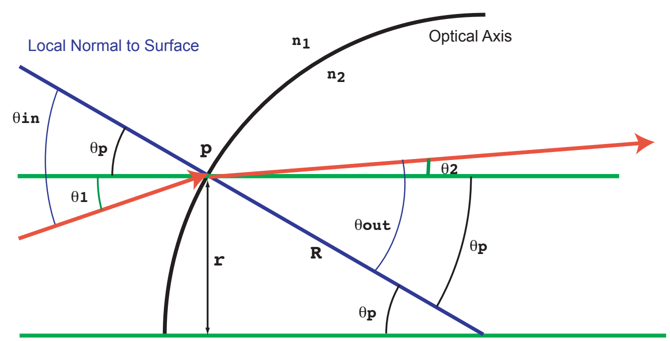
Ray Matrices

ABCD Matrix for curved interface



$\sin(\theta_p) \cong \theta_p \cong r/R$

$r'_1 = \tan(\theta_1) \cong \theta_1$

$r'_2 = \tan(\theta_2) \cong \theta_2$

$\theta_{in} = \theta_p + \theta_1 \cong r/R + r'_1$

Snell's Law : $n_1 \sin(\theta_{in}) = n_2 \sin(\theta_{out})$

=> $n_1 \theta_{in} \cong n_2 \theta_{out}$

=> $\theta_{out} \cong (n_1/n_2)\,\theta_{in}$

$\theta_2 = \theta_{out} - \theta_p$

$$\begin{aligned}
\theta_2 &\cong (n_1/n_2)\,\theta_{in} - (r/R) \\
&\cong (n_1/n_2)\,[(r/R) + r'_1] - (r/R) \\
&\cong [(n_1/n_2) - 1]\,(r_1/R) + (n_1/n_2)\,r'_1
\end{aligned}$$

$r'_2 \cong [(n_1 - n_2)/(n_2 R)]\; r_1 + (n_1/n_2)\, r'_1$

Spherical Interface $= \begin{bmatrix} 1 & 0 \\ \dfrac{n1-n2}{n2\ R} & \dfrac{n1}{n2} \end{bmatrix}$

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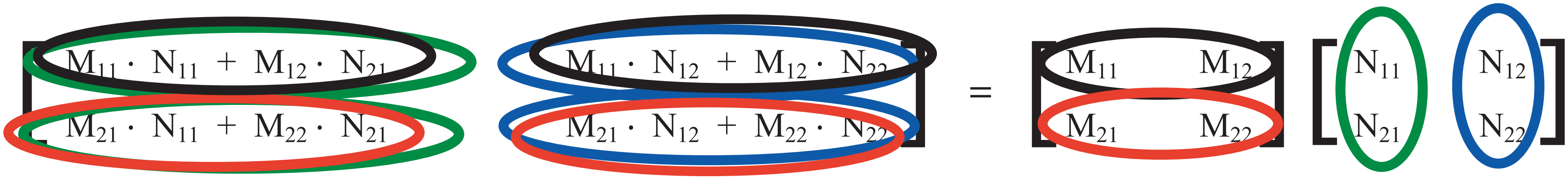
Ray Matrices

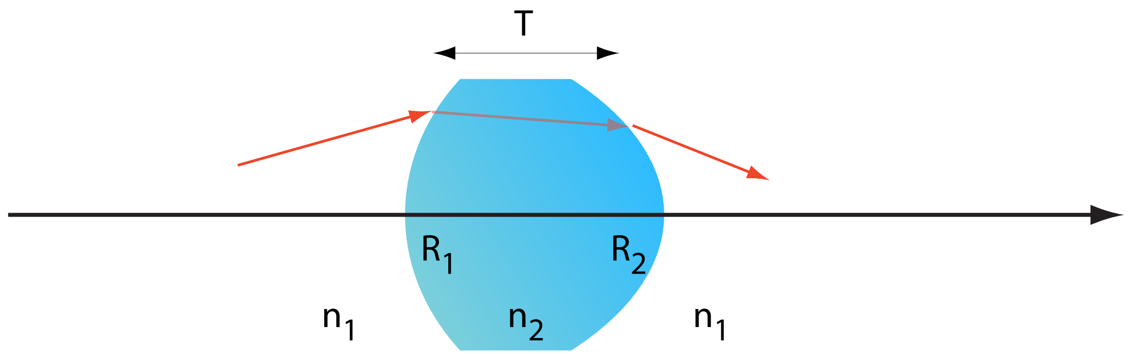
Review of Matrix Algebra

$$\begin{bmatrix} M_{11} \cdot N_{11} + M_{12} \cdot N_{21} & M_{11} \cdot N_{12} + M_{12} \cdot N_{22} \\ M_{21} \cdot N_{11} + M_{22} \cdot N_{21} & M_{21} \cdot N_{12} + M_{22} \cdot N_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$



Ray Matrices

ABCD Matrix for a thick lens

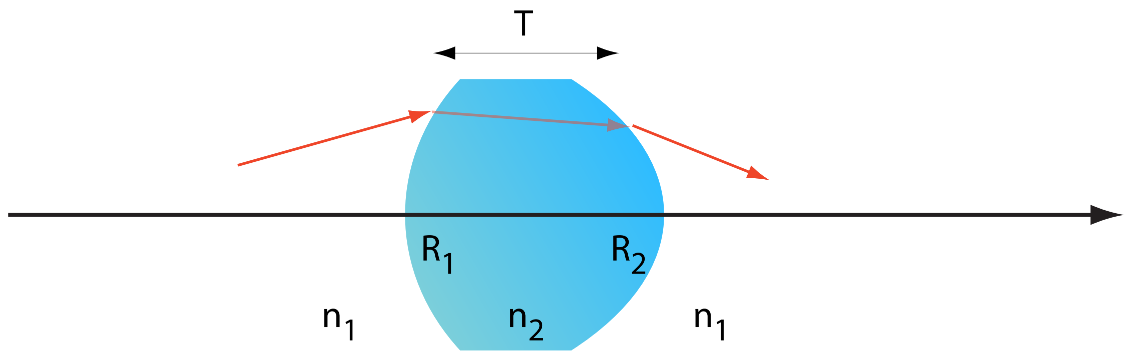


Curved Interface #2 — Propagation through Glass — Curved Interface #1

$$\begin{bmatrix} 1 & 0 \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2} \end{bmatrix}$$

Ray Matrices

ABCD Matrix for a thick lens



Curved Interface #2 | Propagation through Glass | Curved Interface #1

$$\begin{bmatrix} 1 & 0 \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & T \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} & \left(\frac{n_2}{n_1} - 1\right)\frac{T}{R_2} + \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \left(\frac{n_1}{n_2} - 1\right)\frac{T}{R_1} & T\,\frac{n_1}{n_2} \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} + \left(\frac{n_2}{n_1} - 1\right)\left(\frac{n_1}{n_2} - 1\right)\frac{T}{R_1 * R_2} + \frac{n_2}{n_1}\left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2}\left(\frac{n_2}{n_1} - 1\right)\frac{T}{R_2} + 1 \end{bmatrix}$$

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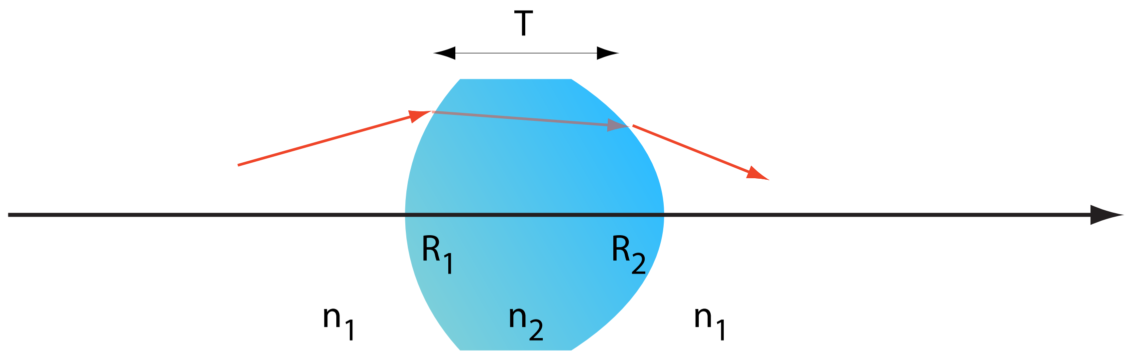
Ray Matrices

ABCD Matrix for a thick lens



Curved Interface #2 — Propagation through Glass — Curved Interface #1

$$\begin{bmatrix} 1 & 0 \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & T \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} & \left(\frac{n_2}{n_1} - 1\right)\frac{T}{R_2} + \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \left(\frac{n_1}{n_2} - 1\right)\cancel{\frac{T}{R_1}} & \cancel{T}\,\frac{n_1}{n_2} \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} + \left(\frac{n_2}{n_1} - 1\right)\left(\frac{n_1}{n_2} - 1\right)\cancel{\frac{T}{R_1 * R_2}} + \frac{n_2}{n_1}\left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2}\left(\frac{n_2}{n_1} - 1\right)\cancel{\frac{T}{R_2}} + 1 \end{bmatrix}$$

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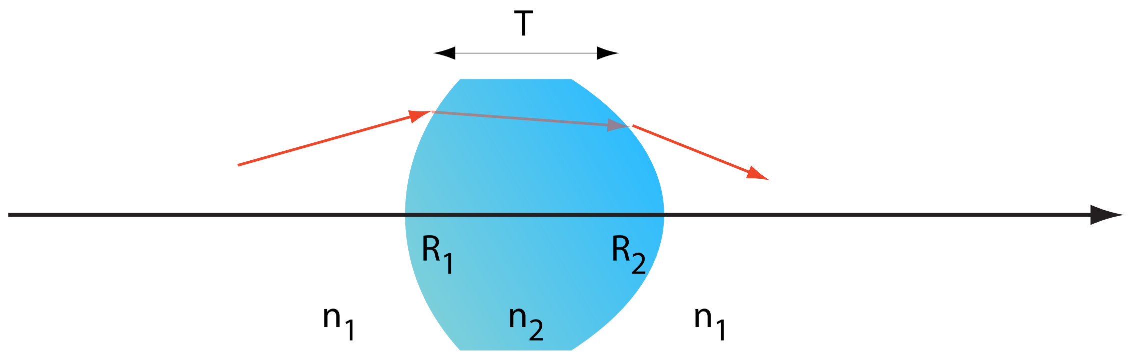
Ray Matrices

ABCD Matrix for a thick lens



Curved Interface #2 — Propagation through Glass — Curved Interface #1

$$\begin{bmatrix} 1 & 0 \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & T \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} & \left(\frac{n_2}{n_1} - 1\right)\frac{T}{R_2} + \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \left(\frac{n_1}{n_2} - 1\right)\frac{\cancel{T}}{R_1} & \cancel{T}\frac{n_1}{n_2} \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} + \left(\frac{n_2}{n_1} - 1\right)\left(\frac{n_1}{n_2} - 1\right)\frac{\cancel{T}}{R_1 * R_2} + \frac{n_2}{n_1}\left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & \frac{n_1}{n_2}\left(\frac{n_2}{n_1} - 1\right)\frac{\cancel{T}}{R_2} + 1 \end{bmatrix}$$

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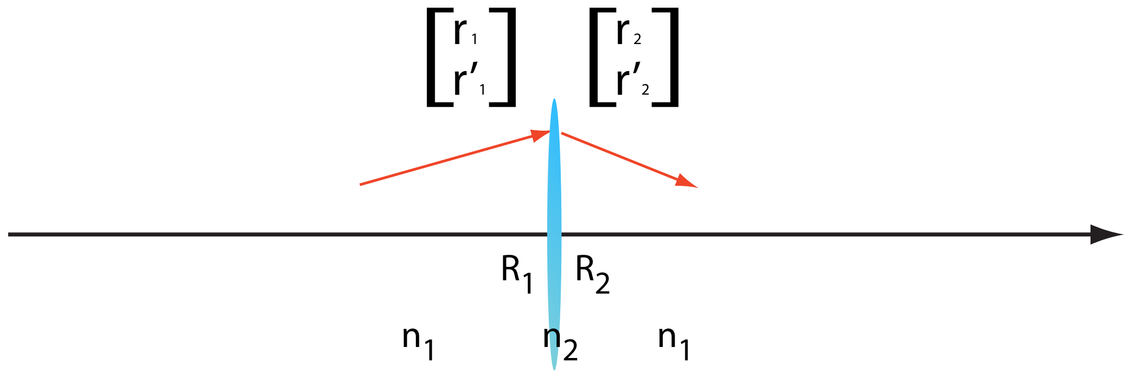
Ray Matrices

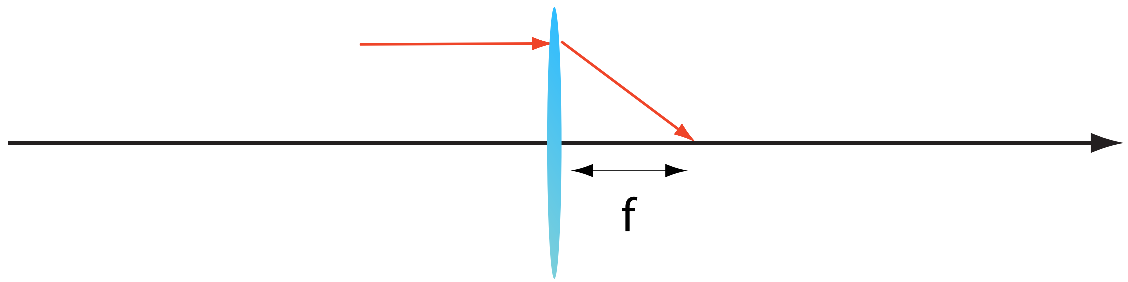
ABCD Matrix for a thin lens



$$\begin{bmatrix} 1 & 0 \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} + \frac{n_2}{n_1}\left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & 1 \end{bmatrix}_{\text{Thin Lens}}$$

Ray Matrices

ABCD Matrix for a thin lens

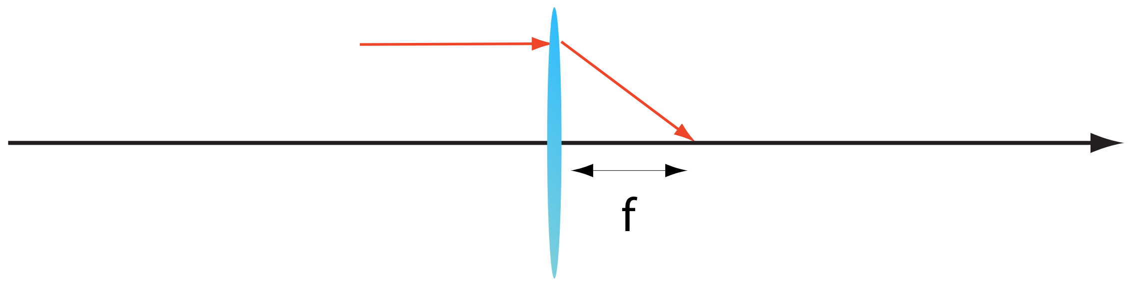


Propagation · Thin Lens · Collimated Input

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_1 \\ 0 \end{bmatrix}$$

Ray Matrices

ABCD Matrix for a thin lens



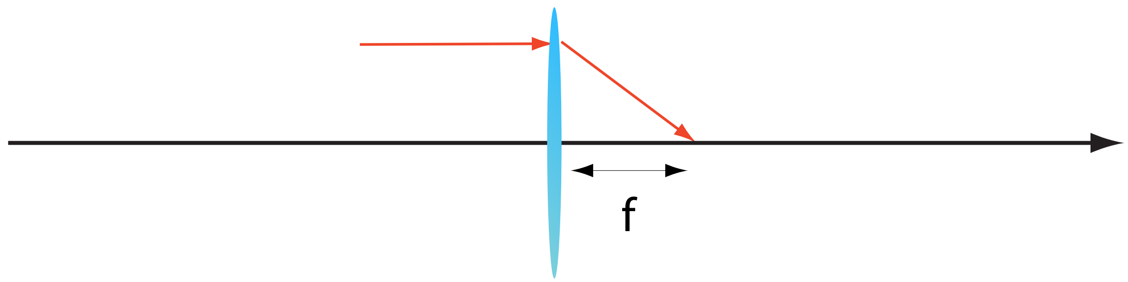
$$\underset{}{\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix}} = \underset{\text{Propagation}}{\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}} \underset{\text{Thin Lens}}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} \underset{\text{Collimated Input}}{\begin{bmatrix} r_1 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \cdot r_1 \\ C \cdot r_1 \end{bmatrix}$$

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} A \cdot r_1 + f \cdot C \cdot r_1 \\ C \cdot r_1 \end{bmatrix}$$

Ray Matrices

ABCD Matrix for a thin lens



Propagation · Thin Lens · Collimated Input

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \cdot r_1 \\ C \cdot r_1 \end{bmatrix}$$

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} A \cdot r_1 + f \cdot C \cdot r_1 \\ C \cdot r_1 \end{bmatrix}$$

$$A \cdot r_1 + f \cdot C \cdot r_1 = 0$$

$A = 1$ (infinitely thin lens)

$\Rightarrow C = -1/f$

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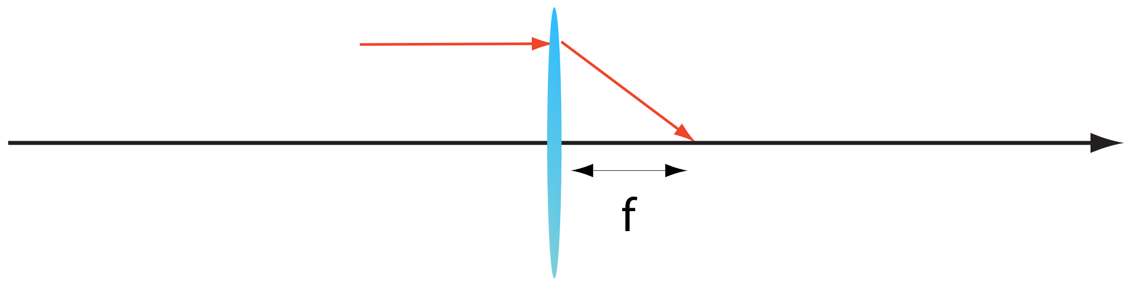
Ray Matrices

ABCD Matrix for a thin lens



Propagation, Thin Lens, Collimated Input

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \cdot r_1 \\ C \cdot r_1 \end{bmatrix}$$

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} A \cdot r_1 + f \cdot C \cdot r_1 \\ C \cdot r_1 \end{bmatrix}$$

$$A \cdot r_1 + f \cdot C \cdot r_1 = 0$$

$A = 1$ (infinitely thin lens)

=> $C = -1/f$

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$ Thin Lens

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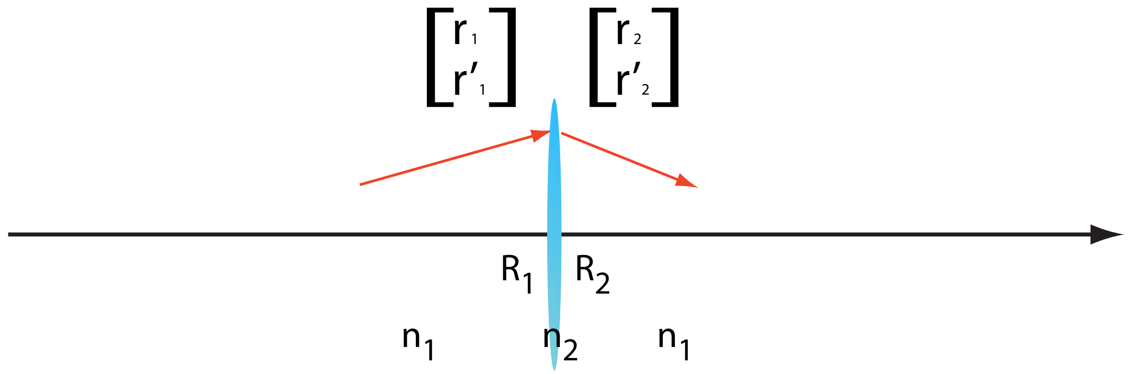
Ray Matrices

ABCD Matrix for a thin lens

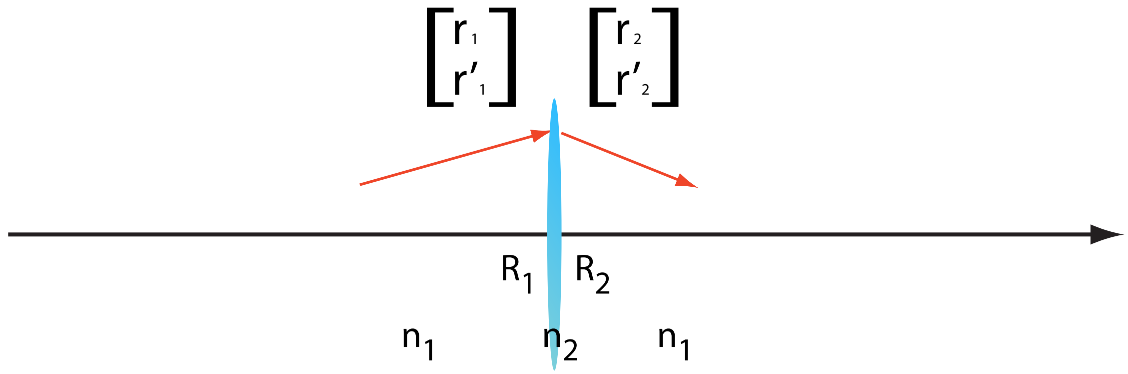


$$\begin{bmatrix} 1 & 0 \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} + \frac{n_2}{n_1}\left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & 1 \end{bmatrix}_{\text{Thin Lens}}$$

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}_{\text{Thin Lens}}$$

Ray Matrices

ABCD Matrix for a thin lens



$$\begin{bmatrix} 1 & 0 \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} + \frac{n_2}{n_1}\left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1} & 1 \end{bmatrix}_{\text{Thin Lens}}$$

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}_{\text{Thin Lens}}$$

$$-1/f = \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R_2} + \frac{n_2}{n_1}\left(\frac{n_1}{n_2} - 1\right)\frac{1}{R_1}$$

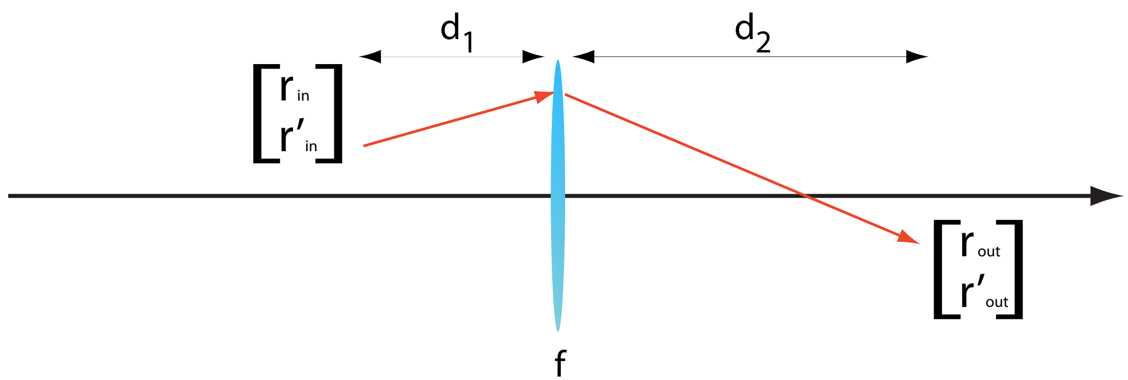
$$-1/f = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

$$1/f = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Lensmaker Formula

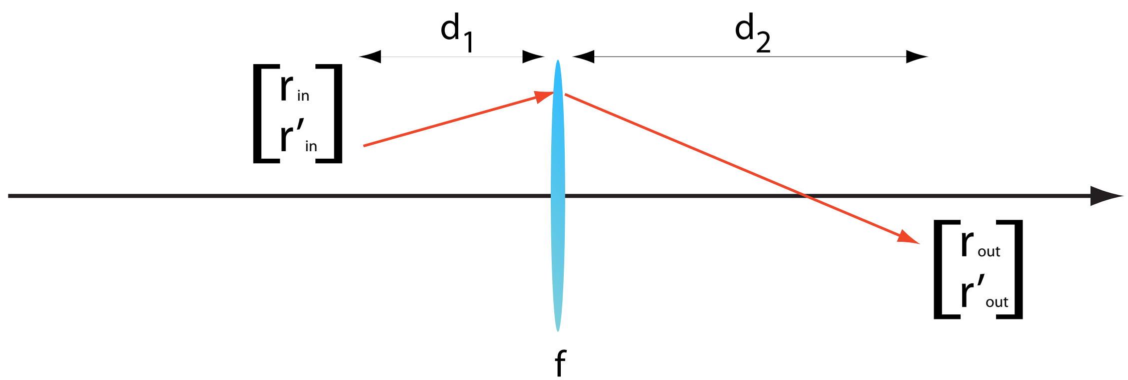
Ray Matrices

Space - Lens - Space



Ray Matrices

Space - Lens - Space



Propgataion d2	Thin Lens (f)	Propagation d1
$\begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix}$

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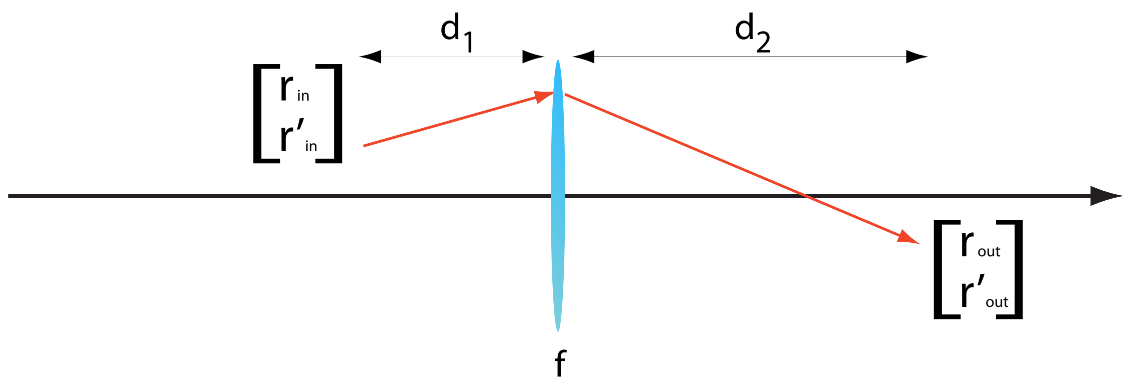
Ray Matrices

Space - Lens - Space



Propgataion d2 — Thin Lens (f) — Propagation d1

$$\begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ -1/f & 1 - d_1/f \end{bmatrix}$$

$$= \begin{bmatrix} 1 - d_2 / f & d_1 + d_2 - (d_1 * d_2) / f \\ -1/f & 1 - d_1/f \end{bmatrix}$$

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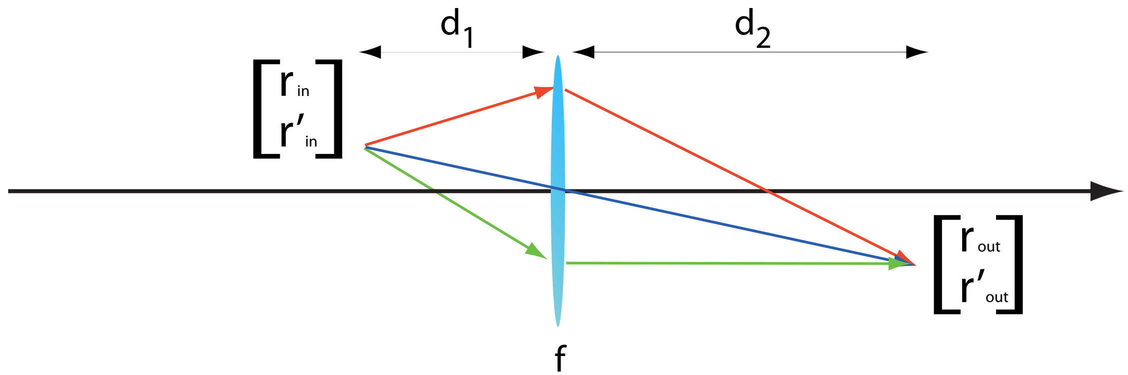
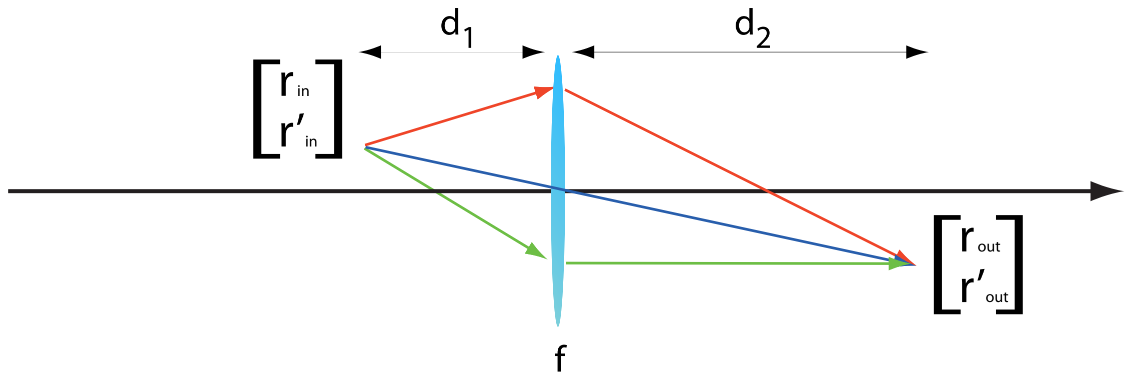
Ray Matrices

Space - Lens - Space



Ray Matrices

Space - Lens - Space



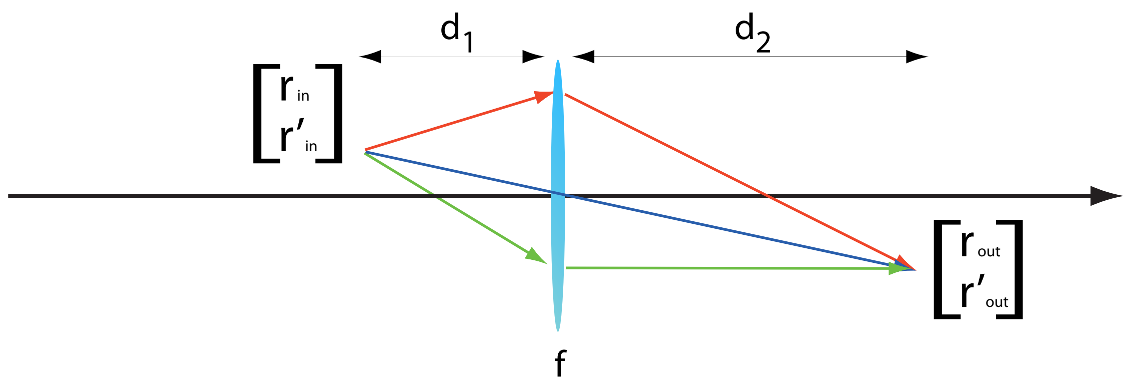
$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_1 \\ r'_1 \end{bmatrix}$$

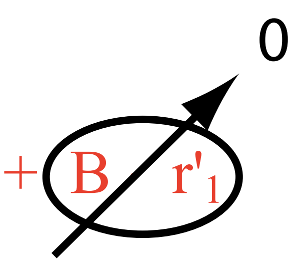
$$r_2 = A \cdot r_1 + B \cdot r'_1$$

$$r'_2 = C \cdot r_1 + D \cdot r'_1$$

Ray Matrices

Space - Lens - Space



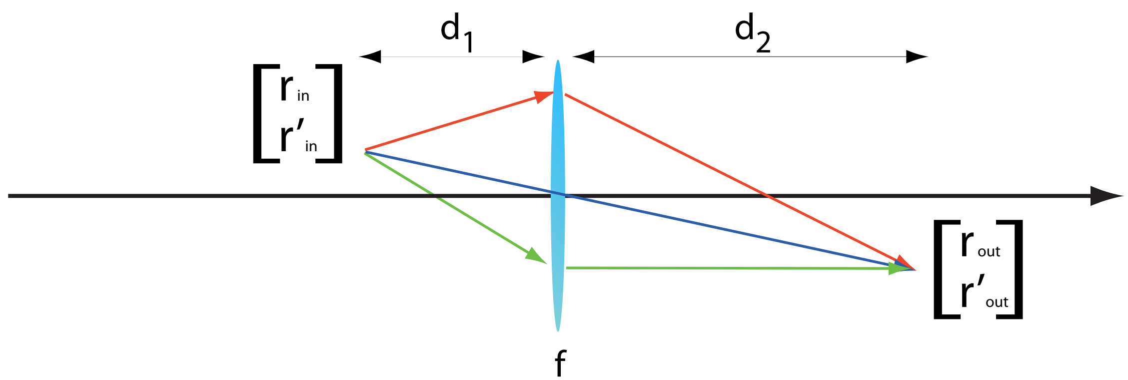


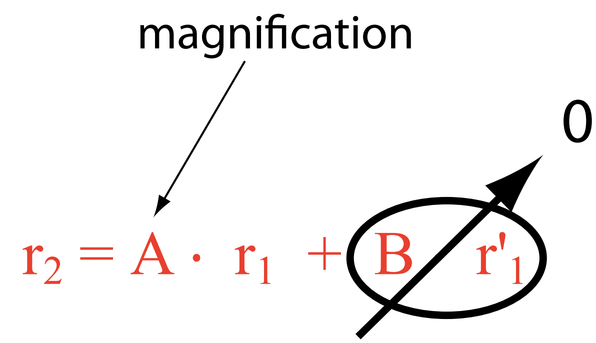
$$r_2 = A \cdot r_1 + B\, r'_1$$

For r_1 to be independent of r'_1
requires : $B = 0$

Ray Matrices

Space - Lens - Space



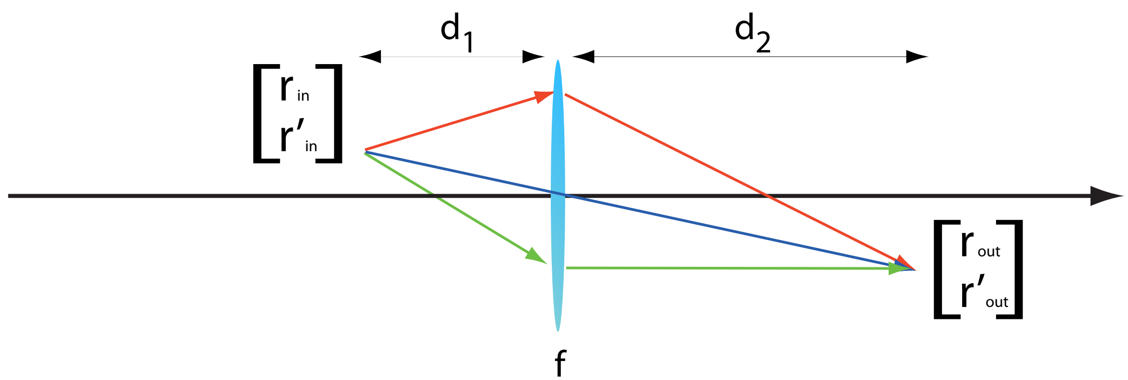


For r_1 to be independent of r'_1
requires : $B = 0$

and yield : A = magnification

Ray Matrices

Space - Lens - Space



$$\begin{bmatrix} 1 - d_2 / f & d_1 + d_2 - (d_1 \cdot d_2) / f \\ -1/f & 1 - d_1/f \end{bmatrix}_{\text{Space - Lens - Space}}$$

magnification

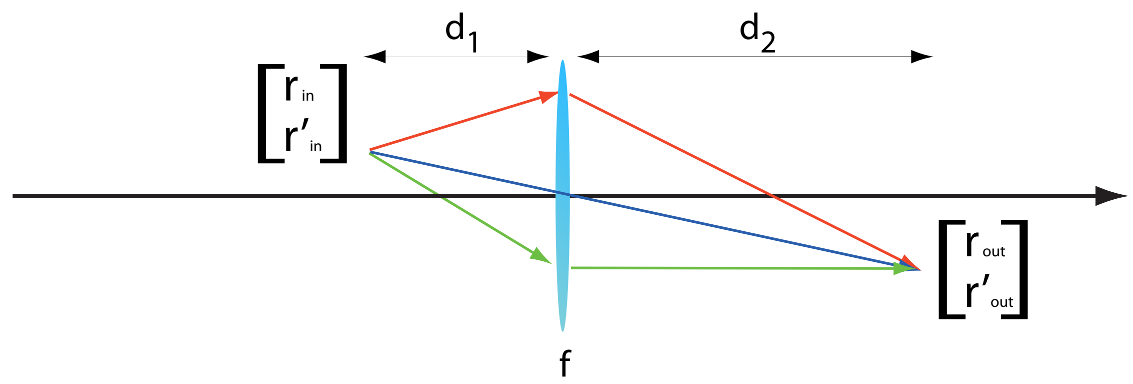
$$r_2 = A \cdot r_1 + B \, r'_1$$

0

Imaging Condition

Ray Matrices

Space - Lens - Space



$$\begin{bmatrix} 1 - d_2 / f & d_1 + d_2 - (d_1 \cdot d_2) / f \\ -1/f & 1 - d_1/f \end{bmatrix}_{\text{Space - Lens - Space}}$$

$$d_1 + d_2 - (d_1 \cdot d_2) / f = 0$$

$$d_2 + d_1 = (d_1 \cdot d_2) / f$$

$$^1/_{d_1} + {}^1/_{d_2} = {}^1/_f$$

magnification

$$r_2 = A \cdot r_1 + B \, r'_1$$

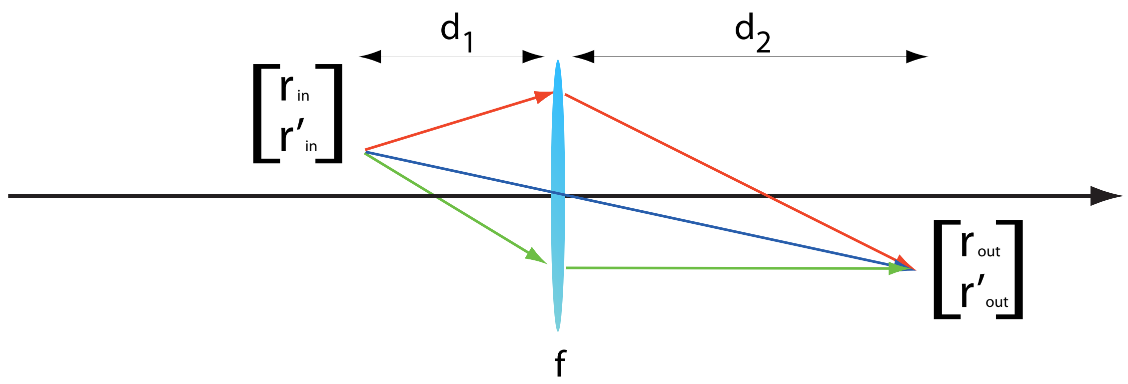
0

Imaging Condition

Philbert S. Tsai - rev. 04/02/2014

Ray Matrices

Space - Lens - Space



$$\begin{bmatrix} 1 - d_2 / f & d_1 + d_2 - (d_1 \cdot d_2) / f \\ -1/f & 1 - d_1/f \end{bmatrix}_{\text{Space - Lens - Space}}$$

$$d_1 + d_2 - (d_1 \cdot d_2) / f = 0$$

$$d_2 + d_1 = (d_1 \cdot d_2) / f$$

$$1/d_1 + 1/d_2 = 1/f$$

$$A = 1 - d_2 \cdot (1/d_1 + 1/d_2)$$

$$= - d_2 / d_1 = \text{magnification}$$

magnification

$$r_2 = A \cdot r_1 + B \cdot r'_1$$

0

Imaging Condition

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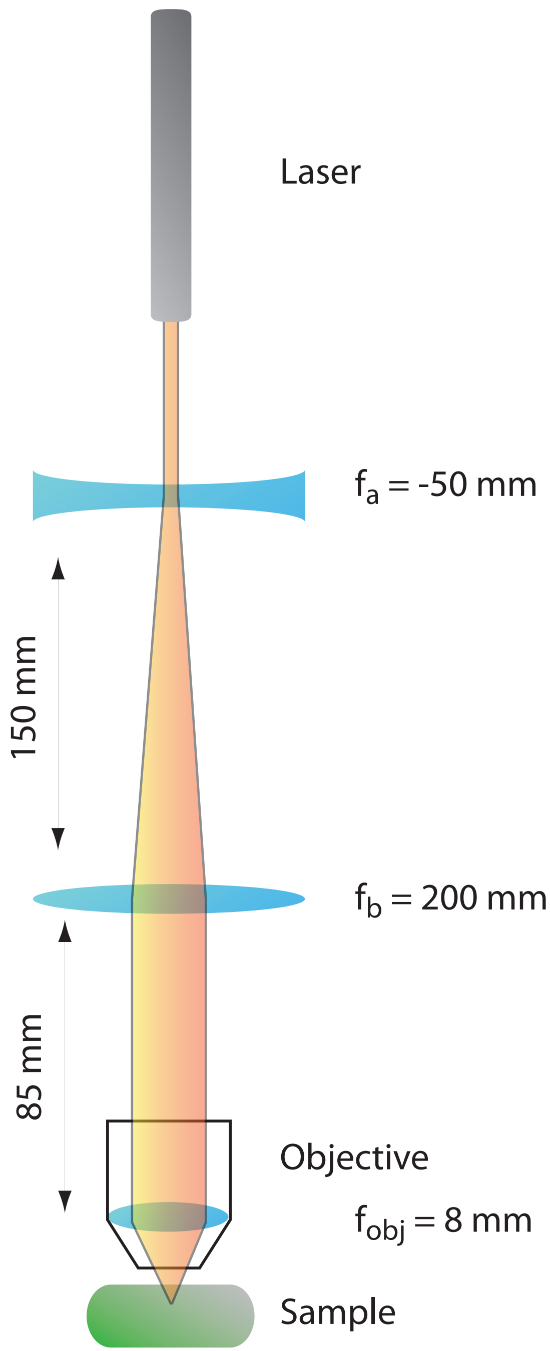
Ray Matrices : a practical application



Problem : Yellow Laser focusing 13 µm deeper than scanning focal plane

MODEL 1

(assume the objective is perfect
but we set the a-b separation poorly)

```
fobj = 8.000
d = 8.013;

sep = 142.1;       <- vary this
M1 = [1 0 ; 1/50 1];
M2 = [1 sep; 0 1];
M3 = [1 0 ; -1/200 1];
M4 = [1 85; 0 1];
M5 = [1 0 ; -1/fobj 1];
M6 = [1 d ; 0 1];

Mtotal = M6*M5*M4*M3*M2*M1;
vo = [1; 0];
vfinal = Mtotal * vo

RESULT:
vfinal =
  -0.0000          <- until this is zero
  -0.4879
```

CONCLUSION:
Move -50mm lens forward
by 7.9 mm

MODEL 2

(assume the objective is imperfect
but the a-b separation is okay)

```
fobj = 8.013
d = 8.000;

sep = 158.3;       <- vary this
M1 = [1 0 ; 1/50 1];
M2 = [1 sep; 0 1];
M3 = [1 0 ; -1/200 1];
M4 = [1 85; 0 1];
M5 = [1 0 ; -1/fobj 1];
M6 = [1 d ; 0 1];

Mtotal = M6*M5*M4*M3*M2*M1;
vo = [1; 0];
vfinal = Mtotal * vo

RESULT:
vfinal =
   0.0000          <- until this is zero
  -0.5119
```

CONCLUSION:
Move -50mm lens forward
by 8.3 mm

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